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Monotone positive solution of a fourth-order BVP with integral boundary conditions

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Abstract

In this paper, we investigate the existence of concave and monotone positive solutions for a nonlinear fourth-order differential equation with integral boundary conditions of the form $x^{(4)}(t) = f(t, x(t), x'(t), x''(t))$, $t \in [0, 1]$, $x(0) = x'(1) = x'''(1) = 0$, $x''(0) = \int_0^1 g(s)x''(s) ds$, where $f \in C([0, 1] \times [0, +\infty)^2 \times (-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$. By using a fixed point theorem of cone expansion and compression of norm type, the existence and nonexistence of concave and monotone positive solutions for the above boundary value problems is obtained. Meanwhile, as applications of our results, some examples are given.

MSC: 34B15; 34B18**Keywords:** fourth-order differential equation; integral boundary condition; monotone positive solution; existence; nonexistence**1 Introduction**

This paper is the follow-up of [1]. In [1], by using a fixed point theorem for the sum of two operators due to O'Regan [2], we obtained existence of solutions for a fully nonlinear fourth-order equation with integral boundary conditions of type

$$\begin{cases} x^{(4)}(t) = f(t, x(t), x'(t), x''(t)), & t \in [0, 1], \\ x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 h(s, x(s), x'(s), x''(s)) ds. \end{cases}$$

In this paper, we study the existence of concave and monotone positive solutions for its simplified form

$$x^{(4)}(t) = f(t, x(t), x'(t), x''(t)), \quad t \in [0, 1] \quad (1.1)$$

subject to the integral boundary conditions

$$\begin{cases} x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 g(s)x''(s) ds, \end{cases} \quad (1.2)$$

where $f \in C([0, 1] \times [0, +\infty)^2 \times (-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$.

It is well known that fourth-order boundary value problems models bending equilibria of elastic beams, and have been studied extensively. Among a substantial number of works dealing with fourth-order boundary value problems, we mention [1, 3–31]. We notice that if $g(\cdot) \equiv 0$ in (1.2), the models are known as the one endpoint simply supported and the other one sliding clamped beam. The study of this class of problems was considered by some authors via various methods, we refer the reader to [4, 7, 10, 14, 15, 23, 26].

The aim of this paper is to establish the existence and nonexistence results of concave and monotone positive solutions for the problems (1.1), (1.2). Here, a solution $x(t)$ of the BVP (1.1), (1.2) is said to be monotone and positive if $x'(t) \geq 0$ on $[0, 1]$ and $x(t) > 0$ on $t \in (0, 1]$. Our main tool is the fixed point theorem of cone expansion and compression of norm type [32]. The paper [33] motivated our study.

2 Preliminary

In this section, we present some lemmas which are needed for our main results.

Throughout this paper, we assume that $f : [0, 1] \times [0, +\infty)^2 \times (-\infty, 0] \rightarrow [0, +\infty)$ and $g : [0, 1] \rightarrow [0, +\infty)$ are continuous, moreover, $\mu := \int_0^1 g(s) ds < 1$.

Simple computations lead to the following lemma.

Lemma 2.1 *For any $h \in C[0, 1]$, the BVP*

$$\begin{cases} x^{(4)}(t) = h(t), & t \in [0, 1], \\ x(0) = x'(1) = x'''(1) = 0, \\ x''(0) = \int_0^1 g(s)x''(s) ds, \end{cases} \quad (2.1)$$

has a unique solution

$$x(t) = \int_0^1 \left[G_1(t, s) + \frac{2t - t^2}{2(1 - \mu)} \int_0^1 G_2(\tau, s)g(\tau) d\tau \right] h(s) ds,$$

where

$$G_1(t, s) = \begin{cases} t(s - \frac{1}{2}s^2) - \frac{1}{6}t^3, & 0 \leq t \leq s \leq 1, \\ s(t - \frac{1}{2}t^2) - \frac{1}{6}s^3, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t, s) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1. \end{cases}$$

Lemma 2.2 *Let $G_1(t, s)$ be as in Lemma 2.1. Then*

$$G_1(t, s) \geq \frac{1}{2} \left(t - \frac{1}{2}t^2 \right) s, \quad (t, s) \in [0, 1] \times [0, 1].$$

Proof For $0 \leq t \leq s \leq 1$, one has

$$\begin{aligned} G_1(t, s) &= t \left(s - \frac{1}{2}s^2 \right) - \frac{1}{6}t^3 \\ &\geq ts \left(1 - \frac{1}{2}s \right) - \frac{1}{6}t^2s \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}ts - \frac{1}{6}t^2s \\ &\geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right)s. \end{aligned}$$

On the other hand, for $0 \leq s \leq t \leq 1$, we have $\frac{1}{6}s^2 + \frac{1}{6}t^2 \leq \frac{1}{3}t$, and then

$$\begin{aligned} G_1(t, s) &= s\left(t - \frac{1}{2}t^2\right) - \frac{1}{6}s^3 \\ &\geq s\left(t - \frac{1}{2}t^2\right) - \left(\frac{1}{3}t - \frac{1}{6}t^2\right)s \\ &= \frac{2}{3}\left(t - \frac{1}{2}t^2\right)s \\ &\geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right)s. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 2.3 *If $h \in C[0, 1]$ with $h(t) \geq 0$ on $[0, 1]$, then the unique solution $x = x(t)$ of the BVP (2.1) satisfies:*

- (1) $x(t) \geq 0$ for $t \in [0, 1]$;
- (2) $x'(t) \geq 0$, $x''(t) \leq 0$ for $t \in [0, 1]$, and

$$x(t) \geq \frac{1}{2}\left(t - \frac{1}{2}t^2\right)\|x''\|_{\infty}, \quad t \in [0, 1].$$

Proof (1) From Lemma 2.2 and the fact

$$ts \leq G_2(t, s) \leq s, \quad \forall (t, s) \in [0, 1] \times [0, 1],$$

it follows that

$$x(t) = \int_0^1 \left[G_1(t, s) + \frac{2t - t^2}{2(1 - \mu)} \int_0^1 G_2(\tau, s)g(\tau) d\tau \right] h(s) ds \geq 0, \quad t \in [0, 1]. \quad (2.2)$$

- (2) Note that whenever $(t, s) \in [0, 1] \times [0, 1]$,

$$\frac{\partial}{\partial t} G_1(t, s) \geq 0, \quad \frac{\partial^2}{\partial t^2} G_1(t, s) = -G_2(t, s) \leq 0,$$

it follows that

$$\begin{aligned} x'(t) &= \int_0^1 \left[\frac{\partial}{\partial t} G_1(t, s) + \frac{1 - t}{1 - \mu} \int_0^1 G_2(\tau, s)g(\tau) d\tau \right] h(s) ds \geq 0, \quad t \in [0, 1], \\ x''(t) &= \int_0^1 \left[-G_2(t, s) - \frac{1}{1 - \mu} \int_0^1 G_2(\tau, s)g(\tau) d\tau \right] h(s) ds \leq 0, \quad t \in [0, 1]. \end{aligned} \quad (2.3)$$

On the one hand, by (2.3), we have

$$\|x''\|_{\infty} \leq \int_0^1 \left[s + \frac{1}{1 - \mu} \int_0^1 G_2(\tau, s)g(\tau) d\tau \right] h(s) ds. \quad (2.4)$$

On the other hand, in view of (2.2) and Lemma 2.2, we have

$$\begin{aligned} x(t) &= \int_0^1 \left[G_1(t, s) + \frac{2t - t^2}{2(1 - \mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] h(s) ds \\ &\geq \int_0^1 \left[\frac{1}{2} \left(t - \frac{1}{2} t^2 \right) s + \frac{2t - t^2}{2(1 - \mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] h(s) ds \\ &\geq \frac{1}{2} \left(t - \frac{1}{2} t^2 \right) \int_0^1 \left[s + \frac{1}{1 - \mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] h(s) ds, \quad t \in [0, 1]. \end{aligned} \quad (2.5)$$

It follows from (2.4) and (2.5) that

$$x(t) \geq \frac{1}{2} \left(t - \frac{1}{2} t^2 \right) \|x''\|_{\infty}, \quad t \in [0, 1].$$

This completes the proof of the lemma. \square

Let

$$E = \{x \in C^2[0, 1] : x(0) = x'(1) = 0\}$$

be endowed with the norm $\|x\| = \max_{t \in [0, 1]} |x''(t)| =: \|x''\|_{\infty}$. Then E is a Banach space. If we denote

$$K = \left\{ x \in E : x(t) \geq \frac{1}{2} \left(t - \frac{1}{2} t^2 \right) \|x\|, x'(t) \geq 0, x''(t) \leq 0, t \in [0, 1] \right\},$$

then it is easy to see that K is a cone in E .

Now, we define an operator T on K as follows: for $x \in K$,

$$(Tx)(t) = \int_0^1 \left[G_1(t, s) + \frac{2t - t^2}{2(1 - \mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds.$$

By Lemma 2.3, we know that $T(K) \subset K$ and if x is a fixed point of T , then x is a concave and monotone positive solution of the BVP (1.1), (1.2).

Lemma 2.4 $T : K \rightarrow K$ is completely continuous.

Proof First, we show that T is continuous. To do this, suppose $x_n, x_0 \in K$ and $\|x_n - x_0\| \rightarrow 0$ ($n \rightarrow \infty$). Then there exists $M_1 > 0$ such that $\|x_0\|, \|x_n\| \leq M_1$ for all $n \in \mathbb{N} = \{1, 2, \dots\}$. Hence from the continuity of f on $[0, 1] \times [0, M_1]^2 \times [-M_1, 0]$, we have

$$f(t, x_n(t), x'_n(t), x''_n(t)) \rightarrow f(t, x_0(t), x'_0(t), x''_0(t)) \quad (n \rightarrow \infty)$$

uniformly on $[0, 1]$. Also, since

$$\begin{aligned} 0 &\leq G_2(t, s) + \frac{1}{1 - \mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \\ &\leq 1 + \frac{1}{1 - \mu} \int_0^1 g(\tau) d\tau \\ &= \frac{1}{1 - \mu}, \quad (t, s) \in [0, 1] \times [0, 1], \end{aligned}$$

we have

$$\begin{aligned}(Tx_n)''(t) &= \int_0^1 \left[-G_2(t,s) - \frac{1}{1-\mu} \int_0^1 G_2(\tau,s)g(\tau) d\tau \right] f(s, x_n(s), x_n'(s), x_n''(s)) ds \\ &\rightarrow \int_0^1 \left[-G_2(t,s) - \frac{1}{1-\mu} \int_0^1 G_2(\tau,s)g(\tau) d\tau \right] f(s, x_0(s), x_0'(s), x_0''(s)) ds \\ &= (Tx_0)''(t) (n \rightarrow \infty) \quad \text{uniformly on } [0,1],\end{aligned}$$

i.e.,

$$\|(Tx_n)'' - (Tx_0)''\|_\infty = \|Tx_n - Tx_0\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore $T : K \rightarrow K$ is continuous.

Next, we prove that T is relatively compact. With this aim, let $D \subset K$ be a bounded set, then there exists a constant $M_2 > 0$ such that $\|x\| \leq M_2$ for all $x \in D$. Suppose that $\{y_n\} \subset T(D)$, there exist $\{x_n\} \subset D$ such that $Tx_n = y_n$. Let

$$M_3 = \sup\{f(t, x_0, x_1, x_2) : (t, x_0, x_1, x_2) \in [0,1] \times [0, M_2]^2 \times [-M_2, 0]\}.$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}|y_n''(t)| &= |(Tx_n)''(t)| \\ &= \int_0^1 \left[G_2(t,s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s)g(\tau) d\tau \right] f(s, x_n(s), x_n'(s), x_n''(s)) ds \\ &\leq M_3 \int_0^1 \left[s + \frac{1}{1-\mu} \int_0^1 sg(\tau) d\tau \right] ds \\ &= \frac{M_3}{2(1-\mu)}, \quad t \in [0,1]\end{aligned}$$

and

$$\begin{aligned}|y_n^{(4)}(t)| &= |(Tx_n)^{(4)}(t)| \\ &= f(t, x_n(t), x_n'(t), x_n''(t)) \\ &\leq M_3, \quad t \in [0,1].\end{aligned}$$

Consequently there exists a constant $M_4 > 0$ such that, for all $n \in \mathbb{N}$,

$$|y_n'''(t)| \leq M_4, \quad t \in [0,1].$$

By the Arzela-Ascoli theorem, we know that $\{y_n''\}$ has a convergent subsequence in supremum norm, i.e., $\{y_n\}$ has a convergent subsequence in E , which indicates that $T(D) \subset K$ is relatively compact in E . This completes the proof of the lemma. \square

The following fixed point theorem of cone expansion and compression of norm type plays a crucial role in our paper.

Lemma 2.5 ([32]) *Let E be a Banach space and let K be a cone in E . Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (i) $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$, or
- (ii) $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

For convenience, firstly we introduce some notations:

$$\begin{aligned} f^0 &= \limsup_{x_0+x_1-x_2 \rightarrow 0^+} \max_{t \in [0,1]} \frac{f(t, x_0, x_1, x_2)}{x_0 + x_1 - x_2}, & f_0 &= \liminf_{x_0+x_1-x_2 \rightarrow 0^+} \min_{t \in [0,1]} \frac{f(t, x_0, x_1, x_2)}{x_0 + x_1 - x_2}, \\ f^\infty &= \limsup_{x_0+x_1-x_2 \rightarrow +\infty} \max_{t \in [0,1]} \frac{f(t, x_0, x_1, x_2)}{x_0 + x_1 - x_2}, & f_\infty &= \liminf_{x_0+x_1-x_2 \rightarrow +\infty} \min_{t \in [0,1]} \frac{f(t, x_0, x_1, x_2)}{x_0 + x_1 - x_2}, \\ H_1 &= \frac{3}{2(1-\mu)}, & H_2 &= \frac{1}{4} \int_0^1 s^2 \left(1 - \frac{1}{2}s\right) \left[\frac{1}{2} + \frac{1}{1-\mu} \int_0^1 \tau g(\tau) d\tau \right] ds. \end{aligned}$$

Theorem 3.1 *If $H_1 f^0 < 1 < H_2 f_\infty$, then the BVP (1.1), (1.2) has at least one concave and monotone positive solution.*

Proof Since $H_1 f^0 < 1$, there exists $\varepsilon_1 > 0$ such that

$$H_1(f^0 + \varepsilon_1) < 1. \quad (3.1)$$

By the definition of f^0 and the continuity of f , there exists $\rho_1 > 0$ such that, for $t \in [0, 1]$, $x_0 + x_1 - x_2 \in [0, \rho_1]$,

$$f(t, x_0, x_1, x_2) < (f^0 + \varepsilon_1)(x_0 + x_1 - x_2). \quad (3.2)$$

Let $\Omega_1 = \{x \in E : \|x\| < \rho/3\}$. For all $x \in K \cap \partial\Omega_1$, from (3.1) and (3.2), we have

$$\begin{aligned} |(Tx)''(t)| &= \int_0^1 \left[G_2(t, s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \\ &\leq \int_0^1 \left[s + \frac{1}{1-\mu} \int_0^1 s g(\tau) d\tau \right] (f^0 + \varepsilon_1) (x(s) + x'(s) - x''(s)) ds \\ &\leq H_1(f^0 + \varepsilon_1) \|x\| \\ &< \|x\|, \quad t \in [0, 1], \end{aligned}$$

which implies that

$$\|Tx\| = \|(Tx)''\|_\infty < \|x\|, \quad \forall x \in K \cap \partial\Omega_1. \quad (3.3)$$

On the other hand, in view of $H_2 f_\infty > 1$, there exists $\varepsilon_2 > 0$ such that

$$H_2(f_\infty - \varepsilon_2) > 1. \quad (3.4)$$

By the definition of f_∞ , there exists $\rho_2 > \rho_1$ such that, for $t \in [0, 1]$, $x_0 + x_1 - x_2 \in [\rho_2, +\infty)$,

$$f(t, x_0, x_1, x_2) > (f_\infty - \varepsilon_1)(x_0 + x_1 - x_2). \quad (3.5)$$

Let $\Omega_2 = \{x \in E : \|x\| < \rho_2\}$. Then for all $x \in K \cap \partial\Omega_2$, from Lemma 2.2, (3.4), and (3.5) it follows that

$$\begin{aligned} (Tx)(1) &= \int_0^1 \left[G_1(1, s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \\ &\geq \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau) d\tau \right] (f_\infty - \varepsilon_2)(x(s) + x'(s) - x''(s)) ds \\ &\geq \frac{1}{2} \int_0^1 \left[\frac{1}{2}s + \frac{s}{1-\mu} \int_0^1 \tau g(\tau) d\tau \right] (f_\infty - \varepsilon_2) \frac{1}{2} \left(s - \frac{1}{2}s^2 \right) \|x\| ds \\ &= H_2(f_\infty - \varepsilon_2) \|x\| \\ &> \|x\|, \quad t \in [0, 1], \end{aligned}$$

which implies that

$$\|Tx\| \geq \|Tx\|_\infty \geq (Tx)(1) > \|x\|, \quad \forall x \in K \cap \partial\Omega_2. \quad (3.6)$$

Therefore, it follows from (3.3), (3.6), and Lemma 2.5 that the operator T has one fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a concave and monotone positive solution of the BVP (1.1), (1.2). This completes the proof of the theorem. \square

Corollary 3.1 Suppose that f is superlinear, i.e.,

$$f^0 = 0, \quad f_\infty = +\infty.$$

Then the BVP (1.1), (1.2) has at least one concave and monotone positive solution.

Theorem 3.2 If $H_1 f^\infty < 1 < H_2 f_0$, then the BVP (1.1), (1.2) has at least one concave and monotone positive solution.

Proof Since $H_2 f_0 > 1$, there exists $\varepsilon_1 > 0$ such that

$$H_2(f_0 - \varepsilon_1) > 1. \quad (3.7)$$

By the definition of f_0 , there exists $\rho_1 > 0$ such that, for $t \in [0, 1]$, $x_0 + x_1 - x_2 \in [0, \rho_1]$,

$$f(t, x_0, x_1, x_2) > (f_0 - \varepsilon_1)(x_0 + x_1 - x_2). \quad (3.8)$$

Let $\Omega_1 = \{x \in E : \|x\| < \rho_1\}$. Then, for all $x \in K \cap \partial\Omega_1$, from Lemma 2.2, (3.7), and (3.8) it follows that

$$\begin{aligned} (Tx)(1) &= \int_0^1 \left[G_1(1, s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \\ &\geq \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau) d\tau \right] (f_0 - \varepsilon_1)(x(s) + x'(s) - x''(s)) ds \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \int_0^1 \left(\frac{1}{2}s + \frac{s}{1-\mu} \int_0^1 \tau g(\tau) d\tau \right) (f_0 - \varepsilon_1) \frac{1}{2} \left(s - \frac{1}{2}s^2 \right) \|x\| ds \\
&= H_2(f_\infty - \varepsilon_1) \|x\| \\
&> \|x\|, \quad t \in [0, 1],
\end{aligned}$$

which implies that

$$\|Tx\| > \|x\|, \quad \forall x \in K \cap \partial\Omega_1. \quad (3.9)$$

On the other hand, in view of $H_1 f^\infty < 1$, there exists $\varepsilon_2 > 0$ such that

$$H_2(f^\infty + \varepsilon_2) < 1. \quad (3.10)$$

By the definition of f^∞ , there exists $\rho^* > 3\rho_1$ such that, for $t \in [0, 1]$, $x_0 + x_1 - x_2 \in [\rho^*, +\infty)$,

$$f(t, x_0, x_1, x_2) < (f^\infty + \varepsilon_2)(x_0 + x_1 - x_2).$$

Let

$$\beta = \max \{f(t, x_0, x_1, x_2) : (t, x_0, x_1, x_2) \in [0, 1] \times [0, \rho^*]^2 \times [-\rho^*, 0]\}.$$

Then for $(t, x_0, x_1, x_2) \in [0, 1] \times [0, +\infty)^2 \times (-\infty, 0]$ one has

$$f(t, x_0, x_1, x_2) < (f^\infty + \varepsilon_2)(x_0 + x_1 - x_2) + \beta. \quad (3.11)$$

Now, we choose $\rho_2 > \frac{1}{3} \max \{\rho^*, \frac{\beta H_1}{1 - H_1(f^\infty + \varepsilon_2)}\}$ and let

$$\Omega_2 = \{x \in E : \|x\| < \rho_2\}.$$

For all $x \in K \cap \partial\Omega_2$, from (3.10) and (3.11) it follows that

$$\begin{aligned}
|(Tx)''(t)| &\leq \left\| \int_0^1 \left[G_2(t, s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \right\|_\infty \\
&\leq \max_{0 \leq t \leq 1} \int_0^1 \left[G_2(t, s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] ds \|f(s, x(s), x'(s), x''(s))\|_\infty \\
&\leq \int_0^1 \left(s + \frac{s}{1-\mu} \int_0^1 g(\tau) d\tau \right) ds [(f^\infty + \varepsilon_2) \|x(s) + x'(s) - x''(s)\|_\infty + \beta] \\
&\leq H_1(f^\infty + \varepsilon_2) \|x\| + \frac{\beta}{3} H_1 \\
&< \rho_2 = \|x\|, \quad t \in [0, 1],
\end{aligned}$$

which implies that

$$\|Tx\| < \|x\|, \quad \forall x \in K \cap \partial\Omega_2. \quad (3.12)$$

Therefore, it follows from (3.9), (3.12), and Lemma 2.5 that the operator T has one fixed point $x \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is a concave and monotone positive solution of the BVP (1.1), (1.2). This completes the proof of the theorem. \square

Corollary 3.2 Suppose that f is sublinear, i.e.,

$$f_0 = +\infty, \quad f^\infty = 0.$$

Then the BVP (1.1), (1.2) has at least one concave and monotone positive solution.

Theorem 3.3 Suppose that

$$H_1 f(t, x_0, x_1, x_2) < x_0 + x_1 - x_2$$

for $(t, x_0, x_1, x_2) \in [0, 1] \times [0, +\infty)^2 \times (-\infty, 0]$ with $x_0 + x_1 - x_2 > 0$. Then the BVP (1.1), (1.2) has no concave and monotone positive solution.

Proof By contradiction, assume that x is a concave and monotone positive solution of the BVP (1.1), (1.2). Then

$$x(t) \geq 0, \quad x'(t) \geq 0, \quad x''(t) \leq 0, \quad t \in [0, 1]$$

and

$$x(t) + x'(t) - x''(t) > 0, \quad t \in (0, 1].$$

Hence

$$\begin{aligned} |x''(t)| &= \int_0^1 \left[G_2(t, s) + \frac{1}{1-\mu} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \\ &\leq \int_0^1 \left(s + \frac{1}{1-\mu} \int_0^1 s g(\tau) d\tau \right) f(s, x(s), x'(s), x''(s)) ds \\ &< \int_0^1 s \left(1 + \frac{\mu}{1-\mu} \right) \frac{1}{H_1} (x(s) + x'(s) - x''(s)) ds \\ &\leq \|x\|, \quad t \in [0, 1], \end{aligned}$$

which implies that

$$\|x\| = \|x''\|_\infty < \|x\|.$$

This is a contradiction. Therefore the BVP (1.1), (1.2) has no concave and monotone positive solution. This completes the proof of the theorem. \square

Theorem 3.4 Suppose that

$$H_2 f(t, x_0, x_1, x_2) > x_0 + x_1 - x_2$$

for $(t, x_0, x_1, x_2) \in [0, 1] \times [0, +\infty)^2 \times (-\infty, 0]$ with $x_0 + x_1 - x_2 > 0$. Then the BVP (1.1), (1.2) has no concave and monotone positive solution.

Proof Suppose on the contrary that x is a concave and monotone positive solution of the BVP (1.1), (1.2). Then from Lemma 2.2 we have

$$\begin{aligned} x(1) &= \int_0^1 \left[G_1(1, s) + \frac{1}{2(1-\mu)} \int_0^1 G_2(\tau, s) g(\tau) d\tau \right] f(s, x(s), x'(s), x''(s)) ds \\ &> \int_0^1 \left[\frac{1}{4}s + \frac{1}{2(1-\mu)} \int_0^1 \tau s g(\tau) d\tau \right] \frac{1}{H_2} (x(s) + x'(s) - x''(s)) ds \\ &\geq \frac{1}{2H_2} \int_0^1 \left(\frac{1}{2}s + \frac{s}{1-\mu} \int_0^1 \tau g(\tau) d\tau \right) \frac{1}{2} \left(s - \frac{1}{2}s^2 \right) \|x\| ds \\ &= \|x\|, \end{aligned}$$

which is a contradiction. This completes the proof of the theorem. \square

Finally, we give some examples to demonstrate applications of our results.

Example 3.1 Consider the fourth-order boundary value problem

$$x^{(4)}(t) = \frac{1}{1+t} \left[\frac{x + x' - x''}{4e^{x+x'-x''}} + \frac{34(x + x' - x'')^2}{1 + x + x' - x''} \right], \quad t \in [0, 1], \quad (3.13)$$

$$x(0) = x'(1) = x'''(1) = 0, \quad x''(0) = \int_0^1 s x''(s) ds. \quad (3.14)$$

Let

$$f(t, x_0, x_1, x_2) = \frac{1}{1+t} \left[\frac{x_0 + x_1 - x_2}{4e^{x_0+x_1-x_2}} + \frac{34(x_0 + x_1 - x_2)^2}{1 + x_0 + x_1 - x_2} \right], \quad g(t) = t.$$

Then $f \in C([0, 1] \times [0, +\infty)^2 \times (-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$, and $\mu = \int_0^1 g(s) ds = \frac{1}{2} < 1$. It is easy to compute that

$$f^0 = \frac{1}{4}, \quad f_\infty = 17, \quad H_1 = 3, \quad H_2 = \frac{35}{576},$$

and hence

$$H_1 f^0 < 1 < H_2 f_\infty.$$

So, it follows from Theorem 3.1 that the BVP (3.13), (3.14) has at least one concave and monotone positive solution.

Example 3.2 Consider the fourth-order boundary value problem

$$x^{(4)}(t) = \frac{1}{1+t} \left[\frac{14(x + x' - x'')}{1 + \ln(1 + x + x' - x'')} + \frac{(x + x' - x'')^2}{8(1 + x + x' - x'')} \right], \quad t \in [0, 1], \quad (3.15)$$

$$x(0) = x'(1) = x'''(1) = 0, \quad x''(0) = 3 \int_0^1 s^3 x''(s) ds. \quad (3.16)$$

Let

$$f(t, x_0, x_1, x_2) = \frac{1}{1+t} \left[\frac{14(x_0 + x_1 - x_2)}{1 + \ln(1 + x_0 + x_1 - x_2)} + \frac{(x_0 + x_1 - x_2)^2}{8(1 + x_0 + x_1 - x_2)} \right], \quad g(t) = 3t^3.$$

Then $f \in C([0, 1] \times [0, +\infty)^2 \times (-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$, and $\mu = \int_0^1 g(s) ds = \frac{3}{4} < 1$. It is easy to compute that

$$f^\infty = \frac{1}{8}, \quad f_0 = 7, \quad H_1 = 6, \quad H_2 = \frac{29}{192},$$

and hence

$$H_1 f^\infty < 1 < H_2 f_0.$$

So, it follows from Theorem 3.2 that the BVP (3.15), (3.16) has at least one concave and monotone positive solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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